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Algebraic partial Boolean algebras

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Abstract
Partial Boolean algebras, first studied by Kochen and Specker in the 1960s, provide the structure for Bell–Kochen–Specker theorems which deny the existence of non-contextual hidden variable theories. In this paper, we study partial Boolean algebras which are 'algebraic' in the sense that their elements have coordinates in an algebraic number field. Several of these algebras have been discussed recently in a debate on the validity of Bell–Kochen–Specker theorems in the context of finite precision measurements.

The main result of this paper is that every algebraic finitely-generated partial Boolean algebra $B(T)$ is finite when the underlying space $\mathcal{H}$ is three-dimensional, answering a question of Kochen and showing that Conway and Kochen's infinite algebraic partial Boolean algebra has minimum dimension. This result contrasts the existence of an infinite (non-algebraic) $B(T)$ generated by eight elements in an abstract orthomodular lattice of height 3. We then initiate a study of higher-dimensional algebraic partial Boolean algebras. First, we describe a restriction on the determinants of the elements of $B(T)$ that are generated by a given set T. We then show that when the generating set T consists of the rays spanning the minimal vectors in a real irreducible root lattice, $B(T)$ is infinite just if that root lattice has an A_5 sublattice. Finally, we characterize the rays of $B(T)$ when T consists of the rays spanning the minimal vectors of the root lattice E_8.

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1. Introduction

As motivation for the study of partial Boolean algebras, we provide a brief summary of the mathematical structure of orthodox (non-relativistic) quantum mechanics. To every physical system there corresponds a Hilbert space $\mathcal{H}$, and any state of this system is described by a unit vector ψ in $\mathcal{H}$. A physical observable of the system is identified with a self-adjoint operator E on $\mathcal{H}$. Its eigenvalues are postulated to be the only possible physical results of measuring E, and an eigenvalue occurs in this context just when the state of the system is in the corresponding eigenspace. Among the observables for a system are the orthogonal projection operators which, in many interpretations of quantum mechanics, are identified with properties of the system. Since every projection can be identified with its range, this establishes a correspondence between quantum-mechanical properties and the set $S(\mathcal{H})$ of closed subspaces in $\mathcal{H}$. For descriptions of the structure of $S(\mathcal{H})$ and the many generalizations that have been proposed to represent the logic of quantum mechanics, see e.g. [5, 7, 8, 15–17].

In this paper we focus on certain substructures in $S(\mathcal{H})$, instances of the partial Boolean algebras of Kochen and Specker [10], which we now motivate and describe. For elements a and b in $S(\mathcal{H})$, let $a \vee b$ be the closure of $a + b$, $a \wedge b$ the intersection of a and b, and a' the orthogonal complement of a in $\mathcal{H}$. Two quantum-mechanical properties are said to be 'simultaneously verifiable' when their corresponding projection operators commute, and this holds just if their eigenspaces satisfy the following geometric relationship. Two subspaces a and b in $S(\mathcal{H})$ are *compatible*, written $a \diamond b$, if $a = x \vee z$ and $b = y \vee z$ for mutually orthogonal subspaces x, y and z in $S(\mathcal{H})$. Algebraically, the condition is equivalent to requiring that a and b be members of a common Boolean subalgebra in $S(\mathcal{H})$ which we now describe.

Extending compatibility to a finite set T of more than two elements, define $B(T)$ to be the partial subalgebra of $S(\mathcal{H})$ generated by T under orthocomplementation $'$ and the partial operations $\wedge$ and $\vee$, the two binary operations being defined only for pairs of elements compatible in $S(\mathcal{H})$. Then $B(T)$ has the structure of a partial Boolean induced by $S(\mathcal{H})$ [1]. In general, a partial Boolean algebra (abbreviated as 'pBa') β consists of a collection $\{\beta_\alpha\}$ of Boolean algebras pasted together in a consistent manner: the binary operations $\wedge$ and $\vee$ are only partial operations in β, defined just when their operands are elements of a common β_α, and in agreement with $\wedge_\alpha$ and $\vee_\alpha$, respectively. pBa provide the simplest physically-meaningful generalizations of mutual compatibility of the elements of T by permitting the existence of elements which belong to different Boolean algebras [11]. Employed in the literature primarily to provide Bell–Kochen–Specker contradictions for hidden variable theories, these algebras are poorly understood for even small sets T (see e.g. [10, 11, 21]).

In this paper, we investigate $B(T)$ in $\mathcal{H} = \mathbb{R}^n$ and $\mathbb{C}^n$ that are 'algebraic' in the sense that the elements of $B(T)$ are the spans of sets of vectors in K^n, where K is a number field (a finite extension of the rational numbers in $\mathbb{C}^n$). A point of departure is Conway and Kochen's example of a pBa in $S(\mathbb{C}^4)$ [11]. It is generated by any five of the six 1-eigenspaces corresponding to the spins in three orthogonal directions x, y and z of two independent spin-$\frac{1}{2}$ particles:

$$\begin{array}{ll} \sigma_x \otimes 1 & 1 \otimes \sigma_x \\ \sigma_y \otimes 1 & 1 \otimes \sigma_y \\ \sigma_z \otimes 1 & 1 \otimes \sigma_z. \end{array}$$

Here, σ_x, σ_y and σ_z are Pauli spin matrices. The pBa is algebraic since its elements are the spans of vectors lying in K^4, where $K = \mathbb{Q}(i)$. It is an infinite algebra; moreover, the group of symmetries of the algebra is dense in the unitary group of $\mathbb{C}^4$.

We first show that every algebraic finitely-generated $B(T)$ is finite when $\mathcal{H}$ is three-dimensional, answering a question of Kochen [11] and showing that Conway and Kochen's infinite algebraic pBa has minimum dimension. This result contrasts the existence of an infinite (non-algebraic) $B(T)$ generated by eight elements in an abstract orthomodular lattice of height 3. We then initiate a study of higher-dimensional algebraic partial Boolean algebras. First we describe a restriction on the determinants of the elements of $B(T)$ that are generated by a given set T. We then show that when the generating set T consists of the rays spanning the minimal vectors in a real irreducible root lattice, $B(T)$ is infinite just if that root lattice has an A_5 sublattice. Finally, we characterize the elements of $B(T)$ when T consists of the rays spanning the minimal vectors of the root lattice E_8.

Recently, there has been a debate involving certain algebraic pBa. Meyer [14] argues that finite precision measurements 'nullify' Bell–Kochen–Specker theorems in $\mathbb{R}^3$, an argument Kent [9] extends to any finite-dimensional real or complex Hilbert space. Subsequently, many authors have defended Bell–Kochen–Specker theorems in these contexts (see e.g. [2, 6, 13]). We do not wish to join this debate, as this is a study of pBa independent of their application to Bell–Kochen–Specker theorems; however, we hope that the algebraic theory developed here can inform those discussions.

1.1. Notation and definitions

Let $S(\mathcal{H})$ be the set of closed subspaces of a Hilbert space $\mathcal{H}$, and let T be a subset of $S(\mathcal{H})$ closed under $'$ and containing 0. Define

$$\Diamond(T) := T \cup \{a \wedge b, a \vee b \mid a \diamond b \text{ for } a, b \in T\}.$$

The set $\Diamond(T)$ is closed under $'$ by de Morgan's laws and the fact that

$$a \diamond b \Leftrightarrow a \diamond b' \Leftrightarrow a' \diamond b \Leftrightarrow a' \diamond b'.$$

Now define the *partial Boolean algebra* $B(T)$ generated by a subset T as

$$B(T) := \bigcup_{n=0}^{\infty} \Diamond^n(T).$$

Here, $\Diamond^n(T) := \Diamond(\Diamond^{n-1}(T))$ for $n \geqslant 1$, where $\Diamond^0(T)$ is defined to be the set obtained from T by adjoining 0 and then closing this set under complementation in $\mathcal{H}$. It will also be useful to define

$$\triangle^n(T) := \Diamond^n(T) \setminus \Diamond^{n-1}(T) \qquad n \geqslant 1$$

as the set of elements 'born on the nth day', and define $\triangle^0(T) := \Diamond^0(T)$.

In discussing specific $\mathcal{H}$, the underlying field is either $\mathbb{R}$ or $\mathbb{C}$. Finite-dimensional $\mathcal{H}$ are equipped with the standard inner product

$$[x, y] = x_1\overline{y_1} + \cdots + x_n\overline{y_n}$$

where $\bar{x}$ represents the complex conjugate of x, and we write $[x]$ for $[x, x]$, the (squared) norm of x.

Finally, we say that $a \in S(\mathcal{H})$ is *algebraic* if $a = \langle v_1, \ldots, v_k\rangle$ with each $v_i \in K$, where K is a number field. A subset $T \subseteq S(\mathcal{H})$ is algebraic if each of its elements is algebraic.

2. Three-dimensional algebraic partial Boolean algebras

For three-dimensional $\mathcal{H}$, every nontrivial element of $S(\mathcal{H})$ is either a ray (a one-dimensional subspace) or the complement of one, so we may refer only to the rays of $S(\mathcal{H})$. We will denote a ray by any spanning vector, remembering that we are only interested in the vector projectively. For distinct rays, compatibility coincides with orthogonality. Thus, with the aforementioned conventions, when $a = \langle v\rangle$ and $b = \langle w\rangle$ we may abbreviate '$a \diamond b$ generate $(a \vee b)'$' to '$v \perp w$ generate vw', where the last product is defined by:

$$\overline{vw} = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1).$$

Our main theorem is

Theorem 2.1. *If* $\dim(\mathcal{H}) = 3$*, then* $B(T)$ *is finite for finite algebraic* T.

Proof. If u and w are two nonzero orthogonal vectors in $\mathbb{C}^3$, then

$$(uw)_i\overline{(uw)_i} = [u][w] - u_i\overline{u_i}[w] - w_i\overline{w_i}[u] \tag{1}$$

and

$$(uw)_j\overline{(uw)_i} = -u_i\overline{u_j}[w] - w_i\overline{w_j}[u] \tag{2}$$

for distinct $i, j \in \{1, 2, 3\}$. Adding the three instances of equation (1) gives the familiar law of composition for the complex numbers:

$$[uw] = [u][w]. \tag{3}$$

This particular proof uses some elementary concepts of algebraic number theory (for an introduction, see [12]). Let $\mathcal{O}_K$ be the number ring of K. Multiplying the vectors in $B(T)$ by appropriate integers in $\mathcal{O}_K$, we can assume that the vectors lie in $\mathcal{O}_K^3$. We can also assume that each vector is primitive, which means that the three coordinates have no common nonunit factor in $\mathcal{O}_K$.

We spend the next few paragraphs showing that $[v]$, where v is any vector in $B(T)$, can have only finitely-many possible values in $\mathcal{O}_K$. We begin by defining, for a finite subset $S \subset \mathcal{O}_K^3$, the set $\Pi(S)$ to consist of all nontrivial prime ideals dividing the norm of at least one member of S. $\Pi(T)$ is finite since T is finite, so let $\Pi(T) = \{\wp_1, \ldots, \wp_t\}$. By equation (3), $\Pi(B(T)) = \Pi(T)$, which we call Π for short.

For each $\wp \in \Pi$, define the 'defect' $\delta_\wp$ to be the minimum integer such that, for each $v \in T$,

$$\wp^\gamma \parallel [v] \Longrightarrow \wp^{\gamma-\delta_\wp} \mid v_i\overline{v_j} \qquad \text{for all} \quad i, j \in \{1, 2, 3\} \tag{4}$$

where $\wp^\gamma \parallel x$ means that the highest power of $\wp$ dividing x is the γth one. Then, in fact, condition (4) holds for all $v \in B(T)$. To see this, suppose that $v \in \triangle^k(T)$. If $k = 0$, v satisfies the condition by the definition of $\delta_\wp$. Otherwise, v is the product of two orthogonal vectors u and w in $\Diamond^{k-1}(T)$ which satisfy the condition by induction. Suppose $\wp^\alpha \parallel [u]$ and $\wp^\beta \parallel [w]$. Then $\wp^{\alpha+\beta} \parallel [uw]$ by equation (3), and $\wp^{\alpha+\beta-\delta_\wp}$ divides each $(uw)_i\overline{(uw)_j}$ by equations (1) and (2), so condition (4) holds for v.

Let h be the order of the ideal class group for $\mathcal{O}_K$ so that $\wp^h$ is principal for any prime ideal $\wp$. Suppose that $\wp^{(2h-1)+\delta_\wp} \mid [v]$ for some $v \in B(T)$. Then $\wp^{2h-1} \mid v_i\overline{v_j}$ for all $i, j \in \{1, 2, 3\}$. Since v is assumed primitive, one of its coordinates is not divisible by $\wp^h$. But this implies that $\wp^h$ must divide each of $\overline{v_1}$, $\overline{v_2}$, and $\overline{v_3}$, which implies that the principal ideal $\bar{\wp}^h$ divides each coordinate of v, contradicting the primitivity of v.

Thus, as an ideal equation,

$$([v]) = \wp_1{}^{\alpha_1}\wp_2{}^{\alpha_2}\ldots\wp_t{}^{\alpha_t}$$

where $0 \leqslant \alpha_i < (2h-1) + \delta_{\wp_i}$ for $i = 1, \ldots, t$. In particular, there are a finite number of possible ideals $([v])$ for primitive $v \in B(T)$, say $(y_1), \ldots, (y_m)$, where each y_i is a real number.

If $([v]) = (y_i)$, then

$$[v] = v_1\overline{v_1} + v_2\overline{v_2} + v_3\overline{v_3} = \mu y_i \tag{5}$$

for some real unit $\mu \in \mathcal{O}_K$. The group of real units in $\mathcal{O}_K$ is finitely-generated, say by $\mu_1, \ldots, \mu_m$. Since we are only concerned with vectors projectively, note that the square of

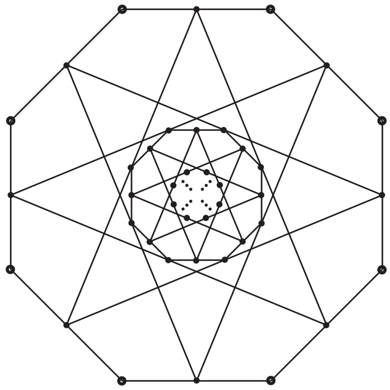

Figure 1. The Greechie diagram for an infinite orthomodular lattice $L = B(T)$, where T contains the eight elements in bold on the outside corners.

any real unit on the right-hand side of equation (5) can be absorbed by the left-hand side. This means that μ can be taken as $\mu = \mu_1^{\epsilon_1}\mu_2^{\epsilon_2}\ldots\mu_m^{\epsilon_m}$ with each $\epsilon_i \in \{0, 1\}$, giving a finite number of possibilities for the right-hand side of equation (5), and thus for $[v]$.

We finish the proof by showing that, for each μy_i, the set of $v \in B(T)$ satisfying equation (5) is finite. Let $n = [K : \mathbb{Q}]$, and let $f_1, \ldots, f_n$ be the embeddings of K in $\mathbb{C}$, where $f_1, \ldots, f_r$ are the real embeddings and $f_{r+2k} = \overline{f_{r+2k-1}}$ are the $n - r$ complex embeddings, $1 \leqslant k \leqslant (n - r)/2$. Applying f_j to equation (5) gives

$$|f_j(v_1)|^2 + |f_j(v_2)|^2 + |f_j(v_3)|^2 = f_j(\mu y_i) \tag{6}$$

so that for each $k \in \{1, 2, 3\}$,

$$|f_j(v_k)|^2 \leqslant f_j(\mu y_i).$$

This means that the set of possible images of v_k in $\mathbb{R}^n$ under the mapping $\theta : K \to \mathbb{R}^n$ given by

$$\theta(u) = (f_1(u), \ldots, f_r(u), \mathrm{Re}(f_{r+1}(u)), \mathrm{Im}(f_{r+2}(u)), \ldots, \mathrm{Re}(f_{n-1}(u)), \mathrm{Im}(f_n(u)))$$

is bounded. Since $\theta(\mathcal{O}_K)$ is an n-dimensional lattice, there are only a finite number of possible values for $v_k \in \mathcal{O}_K$. Thus, there are only a finite number of possibilities for $v = (v_1, v_2, v_3)$. □

We do not know of any non-algebraic finite T in $S(\mathbb{C}^3)$ generating an infinite $B(T)$, and we conjecture that none exists. However, the problem of specializing an arbitrary finite $T \subseteq S(\mathbb{C}^3)$ to the algebraic case appears to be quite hard.

We note that finitely-generated pBa induced by orthomodular lattices of height 3 do exist (for an introduction to the theory of othomodular lattices and posets, see [8]). Figure 1 displays a highly symmetric example of a Greechie diagram corresponding to an infinite $L = B(T)$ when $|T| = 8$. This L, however, need not be embeddable in $S(\mathbb{C}^3)$.

We remark that when P is an orthomodular poset of height 3, the loop lemma restricts its Greechie diagram to containing no loops of order less than 4. It is easy to see that in any orthomodular poset of height 3, $B(T)$ is finite for any subset T containing at most three elements. Figure 2 gives an example of an infinite $P = B(T)$ of height 3 with $|T| = 4$. The construction is related to an independent example of Rogalewicz [19].

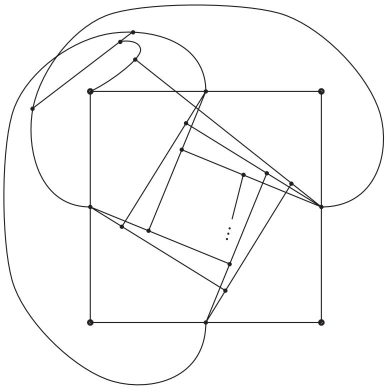

Figure 2. The Greechie diagram for an infinite orthomodular poset $P = B(T)$, where T contains the four elements in bold.

3. Higher-dimensional algebraic partial Boolean algebras

We now turn our attention to algebraic partial Boolean algebras for $\mathcal{H}$ of dimension higher than 3. In particular, we study $B(T)$ for T consisting of spans of sublattices of a given integral lattice L in $\mathbb{R}^n$. In this context, a lattice is *integral* when the inner product of any two of its vectors lies in $\mathbb{Z}$; for an introduction to integral lattices, see [3]. Lemma 3.1 restricts the type of sublattices generated by T, while other conditions ensure the generation of infinite algebras, and we end with an example of an infinite $B(T)$ in $\mathbb{R}^8$ which contains all rays not excluded by lemma 3.1.

To simplify the discussion, we speak of sublattices when technically we mean their spans in $\mathbb{R}^n$, and we refer to specific sublattices by unadorned rows of basis vectors. We use the symbols + and − to represent 1 and −1, respectively, and for coordinates of vectors in $\mathbb{R}^n$, we use an overbar to represent negation. For example,

$$\begin{array}{l} \bar{2}{+}{-}000 \\ {+}{+}{-}{-}{-}{-} \\ {-}{-}{+}{-}{-}{-} \end{array}$$

refers to a three-dimensional subspace of $\mathbb{R}^6$ spanned by three mutually orthogonal vectors of norm 6.

Ubiquitous in discussions of integral lattices are the irreducible root lattices A_n ($n \geqslant 1$), D_n ($n \geqslant 4$) and E_n ($6 \leqslant n \leqslant 8$), since, for example, by Witt's theorem [22] the sublattice generated by vectors of norms 1 and 2 in any integral lattice is the direct sum of irreducible root lattices. For a detailed description of these lattices, as well as a discussion of the 'gluing theory' used in the next section (see chapter 4 of [3]).

3.1. A determinant restriction

The following lemma shows that the determinants of elements of $B(T)$ generated by a set T of sublattices in an integral lattice L are restricted. (If L is not integral but is such that the inner product of any two elements is in $(1/m)\mathbb{Z}$, the discussion may be applied to mL, for example.)

If an n-dimensional integral lattice L contains a sublattice

$$L_1 \oplus \cdots \oplus L_k \tag{7}$$

of total dimension n, then the typical vector y of L can be written as

$$y = y_1 + \cdots + y_k \tag{8}$$

where each y_i is in L_i^*, the dual lattice of L_i. Since the addition of a vector in L_i does not alter the role of y_i, each y_i may be taken from a set of representatives, or 'glue vectors', of L_i^*/L_i, the glue group for L_i. We say then informally that L is generated by (7) and a group of glue vectors (8) which have integral inner products and are closed under addition modulo $L_1 \oplus \cdots \oplus L_k$. A component L_i is said to have 'self-glue' in L if one of the glue vectors y has only one y_i nonzero, the condition occurring when there is a vector in L that is not in L_i but lies in the span of the vectors in L_i.

Each glue group $G = L_i^*/L_i$ is Abelian and thus may be written as a direct sum, each summand involving only a single prime p dividing $|G|$. The sum of those involving p is called the 'p-part' G_p of G.

Lemma 3.1. *Consider the following containments among four n-dimensional integral lattices:*

$$\begin{tikzcd}[column sep=small]
& L \arrow[dl, dash] \arrow[dr, dash] & \\
(X_1 \oplus L_2) \oplus L_4 \arrow[dr, dash] & & (L_1 \oplus X_2) \oplus L_4 \arrow[dl, dash] \\
& (L_1 \oplus L_3 \oplus L_2) \oplus L_4 &
\end{tikzcd}$$

If each component sublattice has no self-glue in the lattice directly containing it, then any prime dividing $\det(L_3)$ *divides* $\det(X_1)$ *or* $\det(X_2)$.

Proof. Suppose p is a prime occurring in $\det(L_3)$ but not in $\det(X_1)$ or $\det(X_2)$. Then every vector y_3 in the p-part G_p of L_3^*/L_3 occurs in some glue vector $y_i + y_3$ for $L_i \oplus L_3$ in X_i, $i = 1, 2$. Suppose $x_1 = y_1 + y_3'$ and $x_2 = y_2 + y_3''$ are two such glue vectors, where $y_3', y_3'' \in G_p$. Since $x_1, x_2 \in L$ and since the L_i are mutually orthogonal,

$$[y_3', y_3''] = [y_1 + y_3', y_2 + y_3''] = [x_1, x_2] \in \mathbb{Z}.$$

Thus, every pair of vectors in G_p must have integral inner product. To see that this is impossible, let y_3 be a nonzero vector in G_p. Every vector v in X_1 can be written as $v = v_1 + v_2 + v_3$, with summands from $(L_1^*/L_1) \oplus L_3$, G_p and $(L_3^*/L_3)/G_p$, respectively. $[y_3, v_1] \in \mathbb{Z}$ since $y_3 \perp (L_1^*/L_1)$ and $y_3 \in L_3^*/L_3$; $[y_3, v_2] \in \mathbb{Z}$ by assumption since $v_2 \in G_p$; and $[y_3, v_3] \in \mathbb{Z}$ since the orders of the two vectors in L_3^*/L_3 are relatively prime. So $[y_3, v] \in \mathbb{Z}$, which implies that $y_3 \in X_1^*/X_1$. However, p divides the order of y_3 but not $\det(X_1) = |X_1^*/X_1|$. □

If the elements of $B(T)$ are sublattices of an integral lattice L, since we are only interested in their spans in $\mathbb{R}$ we may assume that they have no self-glue in L. Lemma 3.1 then implies

Theorem 3.2. *If T is a collection of sublattices of an integral lattice L, the set of primes*

$$\{p \mid p | \det(x), x \in B(T)\}$$

is precisely

$$\Pi = \{p \mid p | \det(x), x \in T\} \cup \{p \mid p | \det(L)\}.$$

A	$=$	+0000	E	$=$	+0000 0++00
B	$=$	++000	F	$=$	++000 00++0
C	$=$	++++0	G	$=$	+0000 0++++
D	$=$	+0000 0+000	H	$=$	+−000 ++++0

Figure 3. Types of sublattices occurring in $B(T_{D_5})$.

For if X is a sublattice of L and X' is the sublattice orthogonal to X in L, then $\det(X') | \det(X) \det(L)$. Also, if X_1 and X_2 are sublattices of L and $X_1 \diamond X_2$, then $X_1 \wedge X_2 = L_3$ as in lemma 3.1, and thus $\det(L_3)$ contains only primes that divide $\det(X_1)$ or $\det(X_2)$. Since the elements of $B(T)$ can be expressed in terms of the elements of T and the operations $\wedge$ and $'$, the set of primes is fixed as Π.

3.2. B(T) for irreducible root lattices

In this section, we consider $B(T)$ when T contains the minimal (nonzero) vectors of an irreducible root lattice.

In discussing the root lattices, we use systems of coordinates given in [3]. Specifically,

$$A_n = \{(x_0, \ldots, x_n) \in \mathbb{Z}^{n+1} \mid x_0 + \cdots + x_n = 0\}$$
$$D_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid x_1 + \cdots + x_n \equiv 0(2)\}.$$

With these coordinates, it will be convenient to define the *type* of a sublattice X as the equivalence class of all sublattices obtained from X by coordinate permutations and, for D_n, sign changes as well.

We write T_L to represent the set of minimal vectors of a lattice L in $\mathbb{R}^n$. For the irreducible root lattices we have

Theorem 3.3. *$B(T_L)$ is infinite for all but finitely-many irreducible root lattices L. It is infinite just when L contains an A_5 sublattice.*

Proof. Since $A_n \subseteq A_{n+1}$, $D_n \subseteq D_{n+1}$, $A_4 \subseteq D_5$, $A_5 \subseteq D_6$ and $A_5 \subseteq E_6 \subseteq E_7 \subseteq E_8$, it will be enough to show that $B(T_{D_5})$ is finite and $B(T_{A_5})$ is infinite.

Any two nontrivial compatible subspaces x, y will generate a Boolean algebra $\mathbf{2}^n$, $2 \leqslant n \leqslant 4$, which will contain n atoms (minimal nonzero elements). Figure 4 shows all of the possible atoms of $B(\{x, y\})$ for $x, y \in B(T_{D_5})$ when $x \diamond y$, where x and y are distinct types as given in figure 3. Figure 5 displays the possibilities when x and y are of the same type. X' refers to the type of the orthogonal complement in D_5 of a sublattice of type X, and 0 refers to the zero subspace. In figure 4, entries below the diagonal correspond to $x \wedge y, x' \wedge y'$; entries above the diagonal are $x \wedge y', x' \wedge y$. Multiple possibilities are stacked within a cell, the kth line in cell (i, j) corresponding to the kth line in cell (j, i).

Since every nontrivial element of $S(\mathbb{R}^5)$ is a one- or two-dimensional subspace or the complement of one, since the elements of T_{D_5} are precisely the members of type B, and since the tables are closed, $B(T_{D_5})$ is finite. Generated by the 20 elements of type B, it contains 952 distinct elements.

	A	B	C	D	E	F	G	H
A		A, B	A, C	A, D $A, 0$	A, E $B, 0$	A, F	$C, 0$	A, H
B	$0, E'$		B, C	B, D $B, 0$	B, E B, E $A, 0$	B, F $B, 0$	B, G	B, H $C, 0$
C	$0, G'$	$0, H'$			C, E	C, F $C, 0$	C, G $A, 0$	C, H $B, 0$
D	$0, D$ A, D'	$0, E$ B, D'			D, E A, B A, B	B, B		
E	$0, E$ A, E'	$0, F$ $0, D$ B, E'	$0, H$	$0, B$ A, E B, D		E, F A, B	B, C	E, H A, C
F	$0, F$	$0, E$ B, F'	$0, G$ C, F'	B, E	$0, B$ B, F		F, G A, C	B, C
G	A, G'	$0, H$	$0, F$ C, G'		A, H	$0, C$ C, F		G, H A, B
H	$0, H$	$0, G$ B, H'	$0, E$ C, H'		$0, C$ B, H	B, G	$0, B$ C, H	

Figure 4. Types generated by pairs of distinct types in $B(T_{D_5})$.

A	$0, D', A, A$
B	$0, F', B, B$ $0, D', B, B$
C	$0, F', C, C$
D	$0, A, D, D$ A, D, A, A
E	$0, A, E, E$ A, D, B, B A, F, B, B B, E, A, A
F	$0, A, F, F$ B, E, B, B C, G, C, C
G	A, F, C, C
H	$0, A, H, H$ B, E, C, C C, G, B, B

Figure 5. Types generated by pairs of the same type in $B(T_{D_5})$.

To see that $B(T_{A_5})$ is infinite, first note that the sublattices of type `2+---0` are generated from the minimal ones of type `+-0000` as in figure 6. Now, consider the isometry given in terms of the standard basis of $\mathbb{R}^6$ by $\phi : x \to Mx$, where

$$M = \frac{1}{6}\begin{bmatrix} 2 & 5 & -1 & -1 & -1 & 2 \\ 5 & -1 & 2 & 2 & -1 & -1 \\ -1 & 2 & 5 & -1 & 2 & -1 \\ -1 & 2 & -1 & 5 & 2 & -1 \\ 2 & -1 & -1 & -1 & 5 & 2 \\ -1 & -1 & 2 & 2 & -1 & 5 \end{bmatrix}.$$

One may verify that ϕ fixes `++++++` and maps the 15 elements of T_{A_5} to (the halves of) the vectors on the right-hand side of figure 7. Inductively, $\phi^{k+1}(T_{A_5})$ is contained in $B(\phi^k(T_{A_5})) \subseteq B(T_{A_5})$, since $\phi^k(T_{A_5})$ is geometrically similar to T_{A_5}, $k = 1, 2, \ldots$. Thus, we

$$\begin{matrix}\texttt{+-0000}\\ \texttt{00+-00}\end{matrix} \wedge \begin{matrix}\texttt{+00-00}\\ \texttt{0-+000}\end{matrix} = \texttt{+-+-00}$$

$$\begin{matrix}\texttt{000+-0}\\ \texttt{+00--+}\\ \texttt{++-00-}\end{matrix} \wedge \begin{matrix}\texttt{00+0-0}\\ \texttt{+0-0-+}\\ \texttt{++0-0-}\end{matrix} = \begin{matrix}\texttt{2+---0}\\ \texttt{0+--3}\overline{\texttt{2}}\end{matrix}$$

$$\begin{matrix}\texttt{+0--0+}\\ \texttt{++00--}\end{matrix} \wedge \begin{matrix}\texttt{2+---0}\\ \texttt{0+--3}\overline{\texttt{2}}\end{matrix} = \texttt{2+---0}$$

Figure 6. Generating sublattices of type 2+---0 from T_{A_5}.

$v \in T_{A_5}$		$2\phi(v)$
+-0000	$\rightarrow$	-2--+0
0+-000	$\rightarrow$	2--+0-
00+-00	$\rightarrow$	002$\bar{2}$00
000+-0	$\rightarrow$	0+-+$\bar{2}$+
0000+-	$\rightarrow$	-0+++$\bar{2}$
+0-000	$\rightarrow$	++$\bar{2}$0+-
0+0-00	$\rightarrow$	2-+-0-
00+0-0	$\rightarrow$	0++-$\bar{2}$+
000+0-	$\rightarrow$	-+02--
+00-00	$\rightarrow$	++0$\bar{2}$+-
0+00-0	$\rightarrow$	2000$\bar{2}$0
00+00-	$\rightarrow$	-+20--
+000-0	$\rightarrow$	+2---0
0+000-	$\rightarrow$	+0++-$\bar{2}$
+0000-	$\rightarrow$	02000$\bar{2}$

Figure 7. Images of T_{A_5} under ϕ.

need only show that $\bigcup_{k=0}^{\infty} \phi^k\left(T_{A_5}\right)$ is infinite. This follows from the fact that, for example, on the two-dimensional subspace of $\mathbb{R}^6$ orthogonal to

+++000
000+++
++$\bar{2}$2--
5$\bar{7}$22--

ϕ is a rotation of infinite order. □

3.3. A comprehensive algebra in $\mathbb{R}^8$

In this final section, we provide an example of a partial Boolean algebra in $\mathbb{R}^8$ which contains all possible rays not excluded by lemma 3.1. To this end, call $B(T)$ *comprehensive* in L if it contains all one-dimensional sublattices of L whose determinants contain only primes in $\Pi = \{p \mid p | \det(x), x \in T\}$.

When $n = 4$ or 8, $\mathbb{R}^n$ admits a multiplication which preserves inner products, leading to the real quaternions $\mathbb{H}$ and octonions $\mathbb{O}$, respectively; with $\mathbb{R}$ and $\mathbb{C}$, they comprise the composition algebras (see [4]). In each of these algebras, there is a notion of a set of 'integers' whose properties mimic the set of rational integers in $\mathbb{R}$. Additively, the maximal set of

$$\begin{matrix} \texttt{++000000} \\ \texttt{00++0000} \end{matrix} \wedge \begin{matrix} \texttt{+0+00000} \\ \texttt{0+0+0000} \end{matrix} = \texttt{++++0000}$$

$$\begin{matrix} \texttt{++++0000} \\ \texttt{0000++++} \end{matrix} \wedge \begin{matrix} \texttt{++00++00} \\ \texttt{00++00++} \end{matrix} = \texttt{++++++++}$$

$$\begin{matrix} \texttt{+++00+00} \\ \texttt{+00++-00} \end{matrix} \wedge \begin{matrix} \texttt{++0+00+0} \\ \texttt{+0+0+0-0} \end{matrix} = \texttt{2++++000}$$

Figure 8. Generating small norm vectors in $B(O_1^8)$.

quaternionic integers H^4 forms a scaled copy of the D_4 lattice, and the maximal set of octonion integers O^8 corresponds to a scaled copy of the E_8 lattice. Coordinates for O^8 can be given as $(x_\infty, x_0, \ldots, x_6)$, where each $x_i \in (1/2)\mathbb{Z}$, and where the set of subscripts for which the coordinates are in fact in $\mathbb{Z}$ is one of the following:

$$\begin{matrix} \emptyset & 0124 & 0235 & 0346 & \infty450 & 0561 & \infty602 & \infty013 \\ \infty0123456 & \infty356 & \infty146 & \infty125 & 1236 & \infty234 & 1345 & 2456. \end{matrix}$$

A result in [20, 4] generalizes the work of Rehm [18] to describe the set of left- and right-hand divisors of norm m of any ρ of norm mn in H^4 or O^8. In particular, we conclude that for primitive ρ, these divisor sets are geometrically similar to the set of units. We now use that result to provide an example of a comprehensive $B(T)$ in $\mathbb{R}^8$.

Theorem 3.4. *$B\big(T_{E_8}\big)$ is comprehensive in E_8.*

Proof. Let O_m^8 denote the set of norm-m elements of O_m^8, so that O_1^8 is geometrically similar to T_{E_8}. Since $2O^8$ is an integral lattice and $\det(2O^8) = 2^4$, lemma 3.1 implies that the only possible norms of elements in O_1^8 are 2^k, $k = 0, 1, \ldots$.

To see that $O_{2^k}^8 \subseteq B\big(O_1^8\big)$ for all k, first note that

$$\begin{matrix} \texttt{+++0+000} \\ \texttt{++-0-000} \end{matrix} \wedge \begin{matrix} \texttt{++0+000+} \\ \texttt{++0-000-} \end{matrix} = \texttt{++000000}$$

where each vector on the left-hand side of the equality is in $2O_1^8$. Using the symmetries of the coordinate system for O^8, all sublattices of type ++000000 can be generated. From these elements, the types on the right-hand side of figure 8 are generated. Since these types contain all vectors in O^8 of norm 2 (multiplied by a factor of 2), we find that $O_2^8 \cdot O_1^8 = O_2^8 \subseteq B\big(O_1^8\big)$. Moreover, since multiplication by $\rho \in O_2^8$ is a geometrical similarity, we conclude in a similar way that any set S of vectors geometrically similar to O_1^8 generates $O_2^8 \cdot S$.

We can now prove easily that $O_{2^k}^8 \subseteq B\big(O_1^8\big)$ by induction on k. For $k \geqslant 1$, let $\rho \in O_{2^k}^8$ be primitive. Then the set $R_{2^{k-1}}(\rho)$ of right-hand divisors ρ of norm 2^{k-1} is geometrically similar to O_1^8, and it is contained in $B\big(O_1^8\big)$ by assumption. Thus, $O_2^8 \cdot R_{2^{k-1}}(\rho) \subseteq B(R_{2^{k-1}}(\rho)) \subseteq B\big(O_1^8\big)$. Since every primitive element of $O_{2^{k-1}}^8$ is in $R_{2^{k-1}}(\rho)$ for some primitive ρ, we must have $O_{2^k}^8 = O_2^8 \cdot O_{2^{k-1}}^8 \subseteq B(O_1^8)$. Since $B\big(T_{E_8}\big)$ is geometrically similar to $B\big(O_1^8\big)$, $B\big(T_{E_8}\big)$ is comprehensive in E_8. □

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